DEEPENING THINKING-LIKE PROBLEMS: THE CASE OF TWO STUDENTS

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Pilot findings of a teaching model based on spiral revision, productive practice and a together-and-apart (TAP) strategy is presented. Subsequently the constituent parts of the teaching model are discussed. A deepening-thinking-like example is used to provide insight on implementation of the model. The conceptual framework based on mathematical reasoning is presented and linked to the study. A classroom instance emerging from implementation of the model is narrated. The classroom instance is analyzed and a possible conclusion is offered.

INTRODUCTION

The term often used by the print media in South Africa to describe the state of mathematics education in the country is crisis. Large scale research such as the Trends in International Mathematics and Science Study (TIMSS) (Mullins, Martin, Foy & Arora, 2012) and the Annual National Assessments (ANA's) (DBE, 2012) paints a bleak picture of the mathematical proficiency of South African learners. Also more often than not mathematics teachers have been identified by print media authors as the main protagonists in this 'crisis'. Many of these authors maintain that poor subject matter knowledge of teachers contribute to the dismal performance of students in mathematics. Now although these criticisms are unfair generalizations since they seem to include all South African teachers irrespective of their success in teaching mathematics, it cannot be denied that South Africa has major problems in the teaching and learning of mathematics. A relevant question therefore is what strategies can be used to enhance the mathematical knowledge and cognitive abilities of pre-service teachers?

Shulman (1986) distinguishes between three categories of teacher content knowledge namely subject matter content knowledge, pedagogical content knowledge and curricular content knowledge. Although the organization, composition and characteristics of mathematical content knowledge for teaching has been extensively researched there is no consensus among researchers concerning the mathematics teachers need to know in order to deliver effective teaching (Ball, Hill & Schilling, 2004). Furthermore although research has shown that teachers' mathematical knowledge is significantly related to learner achievement, the nature and extent of that knowledge is not known (Ball, Hill & Rowan, 2005).

We contend it is the complexity involved in the study of the teaching and learning of mathematics that limits us to research narrow areas of mathematics education, since if we attempt to focus research on too many issues at once we will not be able to sufficiently interrogate each of the issues adequately to provide meaningful contributions to the field. Hence the research described in this paper will focus on how a specific adapted teaching approach influences pre-service teachers in terms of presence and range of competencies in relation to specific mathematical activities (Niss, 2003). The aim of the study therefore is to investigate if a teaching model based on spiral revision, deepening thinking-like problems and TAP (Together-and-Apart) strategies do or do not enhance the procedural and conceptual knowledge and ways of mathematical reasoning of pre-service mathematics teachers. The overarching philosophy of the teaching model is that the teacher should become less and the student more during their regular engagements. That is students should do more and communicate more during lessons and the teacher less. The research will not focus on the teaching model, but rather on an issue emerging from the implementation of the teaching model. It will focus on one classroom instance and zero in on the work of two pre-service mathematics teachers. This is part of a larger project investigating meaningful mathematical education for pre-service teachers being trained to teach at the General Education and Training (GET) level. The teaching model, theoretical framework, conceptual underpinnings of the study and a classroom instance will be discussed in the sections that follow.

THE TEACHING MODEL

One of the cornerstones of the model is spiral (or repeated revision) revision which is defined as the recurrent practising of previously covered mathematical work in specified content areas (Julie, 2013). This is linked to the notion of working memory. When students are required to solve mathematical tasks using newly learned procedures or concepts their working memories can easily be overloaded since they must deal with many new elements of information at once. Conversely, if students had some practice in a mathematical content area they can use their existing knowledge structures to make inferences and make connections between well entrenched concepts in their long term memories to solve mathematical problems presented to them (Cronbach & Snow, 1977; Kalyuga, 2007; Durkin, Rittle-Johnson & Star, 2009). The importance of practicing procedural skills to the extent that it becomes part of the long term memory (automatization) cannot be underestimated. A reason that we advance for the need for spiral revision is that students sometimes struggle with mathematics because they have not practised lower level procedural skills to the extent that it became part of their long-term memories.

What we are advocating is that completed work be revised in class on an on-going basis right through the semester. The idea is that tasks that are conceptually and procedurally more demanding are presented to students during each subsequent cycle of the revision process.

Although the majority of revision problems would be restricted to a specified content area or concept (i.e. the problems would require knowledge from only one content area or concept) some problems presented to students would require integrated knowledge. A major problem in the learning of mathematics is that students tend to compartmentalize mathematical knowledge. Ball (1988) argues that mathematics in the school curriculum is presented in compartments and mathematical content is treated as a collection of discrete bits of procedural knowledge. A consequence of this tendency to compartmentalize mathematical knowledge is that the cognitive load required in knowing and using mathematics is considerably increased. Students instructed in this manner will have a low level of knowledge integration and as a result knowledge accessibility will be negatively affected. Consequently the revision that we envisage should also include tasks whose solutions are based on a combination of two or more concepts from different content areas that have been covered previously and should where possible also require the use of flexible procedural knowledge.

Since it is our intention to use spiral revision to enhance the procedural and conceptual knowledge of pre-service teachers we next discuss these essential knowledge components. It is generally accepted in the mathematics community that conceptual and procedural knowledge are essential knowledge components in the learning of mathematics. Hiebert and Lefevre (1986) contend that conceptual knowledge is knowledge that is rich in relationships. This connected web of knowledge is a network in which the linking relationships are as prominent as the discrete pieces of information. Procedural knowledge is knowledge that consists of rules and procedures for solving mathematical problems. Procedural knowledge also consists of knowledge of mathematical symbols and the syntactic conventions for the manipulation of such symbols (Hiebert & Lefevre, 1986; Star, 2005). Star (2005) argues that skilled problem solvers in mathematics are also flexible in their use of known procedures. A student that does not possess such flexible procedural knowledge sometimes will not be able to solve unfamiliar problems where the solution requires the student to use known procedural knowledge. The student will also not be able to produce a maximally efficient solution in the absence of such flexible procedural knowledge. A result therefore of such flexibility is that students that possess such knowledge will have the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. Star (2005)contends that flexible procedural knowledge is deep procedural knowledge and would allow a student that possess such knowledge to use mathematical procedures that would best fit a provided known or novel problem situation.

Hiebert and Lefevre (1986) argue that sound mathematical knowledge includes significant and fundamental cognitive links between procedural and conceptual knowledge. They maintain that for students to be competent in mathematics they need to possess both procedural and conceptual knowledge and cognitive links between these two essential knowledge components. They describe rote learning as learning that produces knowledge that do not include relationships with other knowledge and which is closely tied to the context in which it was learned. The consequence is that knowledge acquired through rote learning can only be accessed and applied in contexts that mirror the context in which it was learned.

In the model spiral revision is used in conjunction with productive practice. Julie (2013) argues that productive practice is a strategy where students are exposed to deepening thinking-like problems. Deepening thinking-like problems are utilized to enrich the conceptual knowledge of students in requisite content areas of the specified mathematics curriculum.

Students entering initial professional teacher education programmes come from diverse backgrounds, with varying proficiency levels in the various disciplines. This diversity is very prominent in mathematics and consequently teaching such students require a teaching approach that incorporates methods that deal with diversity. The Together- and –Apart (TAP) teaching approach is an attempt to deal with such diversity. The goal of TAP according to its authors (Bennie et al, 2000) is to achieve equity in the mathematics classroom by acknowledging learner diversity in some teaching instances and ignoring diversity in other teaching instances.

A DEEPENING THINKING-LIKE EXAMPLE

Problem solving in mathematics require students to view mathematical concepts from different angles. For example a function can be viewed in terms of its operational character, as a process of co-variation (i.e. how the dependent variable co-varies with the independent variable) or as a mathematical object (Boon, Doorman, Drijvers, Gravemeijer & Reed, 2012). An example of such questions is:

- For what values of x will: $\{(\mathbf{2x}; \mathbf{2x-1}); (\mathbf{x^2-3}; \mathbf{3x}), \mathbf{x} \in \mathbb{R}\}$ not be a function?

This question requires the student to think of the function in terms of co-variation, but importantly also to utilize the definition of the function concept to solve the problem and in so doing perhaps enhancing understanding of the function definition. The solution strategy is also based on the exploitation of a known procedure. The student however has to make the connection between the concept and the procedure. This is our objective with deepening thinking-like questions i.e. to enhance and to deepen conceptual knowledge of students and for the student to make cognitive connections between procedural and conceptual knowledge.

The following solution elucidates the above claims:

Functions can be defined in more than one way. For our purposes we will use the following definition:

A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain of the function and the set B is the range of f.

This definition implies the following:

1. *Each element in the domain must be matched with an element in the range.*
2. *Some elements in the range may not be matched with any element in the domain.*
3. *Two or more elements in the domain may be matched with the same element in the range.*
4. *An element in the domain cannot be matched with two different elements in the range.*

Since the question requires us to find those x values that will cause the ordered pairs not to represent a function we utilize statement 4 above. That is we want the first coordinates to be the same and the second coordinates different. What we therefore are attempting to achieve is to make explicit the fact that a first coordinate cannot be matched with two different second coordinates. The student would therefore be forced to really think about what the definition implies. The following solution explicates this. We start by equating the first coordinates i.e.

$$x^2 - 3 = 2x$$

One should then realize that a known procedure could be utilized to solve for x i.e. one could use solution of quadratic equations. That is the student should make a connection between the procedure and the concept.

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

Which provide two possible solutions i.e.

$$x = -1 \; or \; x = 3$$

We have to check which x-value gives the desired result by substituting into the original coordinate pairs i.e.

$x = -1$, yields $\{(-2, -3); (-2, -3)\}$. This is not the desired result.

If $x = 3,$ *then we have* $\{(6,5); (6,9)\}$. Which is the desired result, since we now have the first coordinates the same and the second coordinates different which violates the 4th statement above i.e. that an element in the domain cannot be matched with two different elements in the range.

IMITATIVE AND CREATIVE REASONING

The conceptual framework underpinning the investigation is based on mathematical reasoning. Lithner (2008) has developed a conceptual framework that is concerned with reasoning in mathematics. In this framework reasoning is defined as the line of thought required to produce assertions and reach conclusions in solving mathematical tasks. Reasoning in this framework can either be the thinking processes or the product of thinking processes or both. A line of thought may be classified as reasoning even if it is incorrect. The only proviso is that it makes sense to the thinker.

Lithner (2008) differentiates between two different types of reasoning in mathematics. Imitative reasoning (IR) occurs when a student produces a solution procedure that he/she memorized. Conversely creative reasoning (CR) is reasoning that is characterized by flexibility and novel approaches to mathematical problems (Bergqvist, 2007).

Lithner (2008) distinguishes between two main two main categories of imitative reasoning, namely memorized and algorithmic imitative (AR) reasoning. He asserts that for imitative reasoning to be classified as memorized reasoning it needs to fulfil two conditions. On the one hand the reasoning should be based on recalling a complete answer. On the other hand the implementation strategy should consist of only writing the answer down. A reasoning sequence is classified as algorithmic reasoning (AR) if the reasoning is based on the recall of an algorithm. An algorithm is described as a finite sequence of executable directives which allows one to find a definite result for a given class of problems. A reasoning sequence is classified as algorithmic reasoning if it satisfies two conditions. On the one hand the strategy choice for the reasoning should be to recall an algorithm as a solution. No other reasoning should be required except to implement the algorithm. Bergqvist (2007) claims that in some instances students use algorithmic reasoning to solve mathematical tasks, without any comprehensive understanding of the underlying mathematical concepts.

Bergqvist (2007) describes creative reasoning (CR) as reasoning that is not hindered by fixation and is characterized by flexibility, novel approaches to mathematical problems and well-founded task solutions. If a task is very nearly solvable using imitative reasoning and creative reasoning is only required to modify an algorithm the reasoning required is local creative reasoning (LCR). On the other hand if a task requires mostly creative reasoning then the reasoning involved is classified as global creative reasoning (GCR).

We are of the opinion that in the majority of cases in South Africa the learning process in mathematics is started with imitative reasoning. In this process the learners imitate the written steps of the procedures of solutions to mathematical tasks that they are shown by their instructor. The idea is that the students not only internalize the external manifestations of the thought processes of the teacher (i.e. the written steps or algorithms) but that they also are aware of and understand the ideas that connect the individual steps and the solution process as a whole. Note I am not contending that students internalize ideas from teachers as a mechanical transaction, but that the process of internalization is underpinned by meaning making by the student. We however ideally want our students to move beyond using only imitative reasoning to instances where they use prior knowledge creatively. The question is what pedagogy would aid in such an endeavour?

A CLASSROOM INSTANCE

The following is a narrative of a classroom incident that occurred when the teaching model was piloted by the lead author. The topic under discussion in this lesson was the notion of rationalisation of denominators. In the lesson rationalisation of denominators of examples such as $\frac{12}{\sqrt{18}}$; $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ and $\frac{1}{\sqrt[5]{6}}$ were discussed. After solutions were provided and discussed for these and other similar examples student A posed a verbal question. Since the lead author did not quite understand his question the student was requested to write his question on the white board. The following was his question. "How would one do this one $\frac{a}{\sqrt[n]{b}}$". The lead author was in the process of providing a longwinded explanation when student A interjected and asked if he could supply a solution. The following is his solution:

$$\frac{a}{\sqrt[n]{b}} \times \frac{\sqrt[n]{b^{n-1}}}{\sqrt[n]{b^{n-1}}}$$

$$= \frac{a.\sqrt[n]{b^{n-1}}}{\sqrt[n]{b^{n-1} \cdot b}}$$

In order to rationalize denominators the notion of one in a different form is utilized as well as the fact that one is the identity element for multiplication in the real number system. The provided solution suggests that student A is aware of both of these ideas. The fact that he did not continue the solution beyond this second step perhaps also suggests that he is aware of the fact that after multiplication is effected the exponent under the root should be equal to the order of the root. His question seems to suggest also that he was wondering what would happen in the case of a variable root. Since all the examples discussed in the lesson utilized integer roots his question extends the discussion into a new direction i.e. what would happen in the case of a general root.

It would seem therefore that student A was attempting to widen or extend the domain of this type of task. Mitchelmore (2002) classifies such attempts as empirical extension generalisation. Such a question can be classified as a higher cognitive question since its solution requires local creative reasoning based on flexible procedural knowledge. Student A does not specify any restrictions on the variables a, b, n and neither does he specifies what kind of numbers a, b and n is.

In engagements between students and teacher in the mathematics classroom it is normally the mathematics teacher that provides questions in order to stimulate students' critical thinking. In the above case the converse happened i.e. student A provided a question that stimulated higher order thinking. Since I was curious as to his level of proficiency with this type of problem I presented student A and the class with the following deepening thinking-like follow-up question:

Simplify the following by rationalising the denominator: $\dfrac{1}{\sqrt[n]{a^{\frac{-c}{m^2-m-2}}}}$

It should be noted that in the part under the root $\frac{-c}{m^2-m-2}$ is the exponent of a.

The intention on the one hand was to revise the work covered previously and on the other to determine if student A would recognise that his previous solution strategy need only be modified slightly to solve this question and therefore there is some generalizability in the solution strategy. Student A however was stumped by this question and could not even start. Student B whom I thought to be primarily an imitative reasoner then claimed that he knew how to solve the problem. The following is his solution:

$$\frac{1}{\sqrt[n]{a^{\frac{-c}{m^2-m-2}}}} \times \frac{\sqrt[n]{a^{\frac{c}{m^2-m-2}+n}}}{\sqrt[n]{a^{\frac{c}{m^2-m-2}+n}}}$$

$$= \frac{\sqrt[n]{a^{\frac{c}{m^2-m-2}+n}}}{\sqrt[n]{a^n}}$$

When I asked him how he arrived at his answer the following dialogue ensued.

Lead author: Why did you decide to multiply by $\dfrac{\sqrt[n]{a^{\frac{c}{m^2-m-2}+n}}}{\sqrt[n]{a^{\frac{c}{m^2-m-2}+n}}}$?

Student B: I know that I have to take the additive inverse of the exponent since this is what we have learnt in the first semester when we were doing properties of rational numbers.

Lead author: Why did you add n?

Student B: So that I can end up with a to the power n.

Haylock (1997) asks the question why a person that knows all the mathematics they need to solve a particular problem still fails to solve it. He contends that some reasons for this might be that the persons' mind is set in an inappropriate direction or that the person is adhering rigidly to an approach that does not lead to a solution. Another possibility might be that the problem solver simply does not make a cognitive connection between known prior requisite knowledge and the provided problem.

I think what we ideally want in the teaching and learning of mathematics is that students develop the ability to use prior knowledge not only flexibly, but also to recognise instances and new contexts where prior knowledge can be applied. Student A could not make the connection between his previous strategy and this new problem although in the previous lesson he was applying prior knowledge in a new way to ask a question that required an extension and modification of the reasoning requirements for such tasks. That is student A was utilizing creative reasoning in the previous lesson, but could not sustain this way of reasoning in the subsequent lesson. Student B however who did not in any of our previous engagements exhibit creative reasoning was able to not only make connections between concepts of the previous lesson, but also with concepts of lessons of the previous semester.

CONCLUSION

The question is what if any conclusions can one draw from the above discussion. Perhaps the above described interactions of students A and B is an indication that if we allow and encourage students from diverse backgrounds to grapple with deepening thinking-like problems then the possibility opens up for them to display their creativity and move beyond concrete primarily number-driven examples to more generalised formulations of some mathematical construct. We are however of the opinion that even with the utilization of a teaching model it takes a long time to change significantly the knowledge levels and cognitive abilities of students. However if a student is willing to attempt and to continue to attempt then the student would as a result of learnings gleaned from these attempts enhance their cognitive abilities and knowledge levels in mathematics. Further investigation is required to determine if the above described teaching model augments the willingness of students to attempt.

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